

The Average Projected Area Theorem – Generalization to Higher Dimensions

Zachary Slepian

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1 Introduction

It is well known that for an arbitrary convex solid in three dimensions, the average projected area is one-fourth the surface area. Perhaps less well-known is the rather touching story of how this theorem originated. Karl Schwarzschild, a German astronomer primarily famous for his black hole solution to Einstein's general relativity, proved the theorem near the turn of the twentieth century. However, perhaps due to his untimely death at forty-two in 1916, the theorem was not well-known. So it remained for his son, astronomer Martin Schwarzschild, noted for his contributions to understanding stellar evolution, especially red giant stars, prove the theorem after the second World War, unaware of his father's work (Knapp, personal communication). One can only imagine his surprise to learn from a colleague that his father had already proved it decades earlier! In the present work, I generalize this theorem to higher dimensions.

What is meant by average projected area? There are two equivalent, convenient ways to imagine this. The first is, hold the solid's position fixed, and shine a light beam behind it toward an observer in front of it. Vary the light beam's direction (and consequently the observer's location, since two points define a line and so the observer must move to see the shadow) through all possibilities, measure the area of the shadow cast at each, and average. The second is, hold the light beam (and observer's) position fixed and rotate the solid through all possible orientations, keeping its spatial location fixed. Measure the shadows and average. Either method yields the same result – as one would expect, since they correspond to the same process but observed either in the rest frame of the solid or of the observer.

So we have set out what average projected area is. But why should one care about the average projected area of a convex solid? Applications are many. In astrophysics, dust is along the path light takes from distant stars or galaxies to us, and so extinguishes some of the light. The dust grains are irregularly shaped and oriented, so it is necessary to correct for their average extinction of the incoming light. This theorem provides the most convenient way to do so. The next step is complex numerical simulations; see e.g. Draine 2011.

The theorem also readily allows one to calculate the temperature of dust grains (or asteroids) illumined by a star (assuming the grains or asteroids are convex). The energy received by the dust grain is just proportional to its projected area (projected onto the light source): clearly, it cannot receive energy from any part of its surface not “visible” to the illumining star. However, the particle can radiate energy away proportionally to its surface area. Setting energy absorbed per unit time equal to energy radiated per unit time (i.e. the luminosities L_{abs} and L_{rad}) determines the equilibrium temperature:

$$L_{\text{rad}} = A_S \sigma_{\text{SB}} T_{\text{grain}}^4 = L_{\text{abs}} = \langle A_{\text{proj}} \rangle \sigma_{\text{SB}} T_*^4 \frac{4\pi R_*^2}{4\pi d^2} (1 - a). \quad (1)$$

A_S is the surface area of the grain, σ_{SB} the Stefan-Boltzmann constant, T_{grain} the grain temperature, $\langle A_{\text{proj}} \rangle$ the average projected area of the grain, T_* the star’s temperature, R_* the star’s radius, d the distance between the grain and the star, and a the albedo, a measure of how reflective of light the grain is (the higher a , the less energy the grain absorbs). Clearly, without the ratio $\langle A_{\text{proj}} \rangle / A_S$, it would be impossible to calculate the temperature of the grain.

The average projected area theorem also has more down-to-earth applications. It is a starting point for determining the 3-d shape of micro-particles using a 2-d map of their projected areas (see Vickers & Brown 2001, Brown et al. 2005). The theorem has military applications as well: Saucier (2000) uses it in discussion of simulating debris fragments behind armor and characterizing the penetration potential of projectiles. Further, it is a useful approximation when considering particle transport in 3-d; see Glassner 1995. The theorem also appears as a step in estimating the sphericity of fruits and of polymer beads in gels; see e.g. Houston 1957 and Nussinovitch 2010. Finally, the theorem has applications in ray tracing for computer graphics (Hanrahan 2000).

The main purpose of this paper is to generalize this theorem to higher-dimensional convex solids. I show how to compute (and prove) the analogous factor to one-fourth (in $\langle A_{\text{proj}} \rangle = \frac{1}{4} A_S$, with A_S the surface area and angle brackets denoting average) for an arbitrary dimension: for example, one can use my results to compute the 7-d average projected area of an arbitrary 8-d convex solid. I also touch on the limit of this factor as the dimension becomes infinite.

The paper is structured as follows. For completeness’ sake, I begin by proving the theorem in 3-d; my treatment is original to this paper, but proofs can be found elsewhere, e.g. in Hildebrand 1942. In Section 3, I prove a method of calculating the analogous ratio for higher dimensions that uses hyperhemispheres. Section 4 applies this method to compute the ratio of average projected area to surface area as a function of the dimension of the space, provides a table of values, and examines the limit of the ratio as the dimension becomes infinite. I continue in Section 5 by presenting a simple, memorable recursion relation for this ratio between dimensions d and $d + 1$. In Section 6, I derive from the recursion relation an explicit formula for the ratio in terms of elementary operations (the formula I supply in Section 4 is in terms of gamma functions).

2 Proof in 3-d

Here we prove that for a convex solid in three dimensions, the average projected area is one-fourth the surface area.

Consider an arbitrary convex solid, and imagine an infinitesimal chunk of surface area dA . Let \vec{dA} be the unit vector normal to the surface. Now imagine surrounding the solid with a sphere, and that we, the observer, sit at a point on the surface of that sphere. If the vector \vec{dA} points such that it intersects the hemisphere whose apex is our location, then we will see it. Since this hemisphere's position is uncorrelated with the direction of \vec{dA} for an arbitrary chunk of surface area dA , on average \vec{dA} will point in the right direction for us to see it half the time.

We now restrict our attention to the case where \vec{dA} does point into "our" hemisphere. Consider its projection along the line of sight. Denote the angle between \vec{dA} and the line of sight by θ , and realize that because we have restricted ourselves to a hemisphere, θ goes from zero to $\pi/2$.

We wish to integrate $\cos \theta$ to find its average value, as $\cos \theta$ gives the projection of \vec{dA} onto the line of sight. When so doing, we will use as our Jacobian for spherical coordinates $\sin \theta d\theta$. Using this Jacobian builds in a certain convention about how θ is defined: namely, that it goes from zero to π . (Basically, think of the line of sight as like the z-axis in spherical coordinates, so θ is the usual angle between the z-axis and \vec{dA} .)

And we must ignore the contribution of θ between $\pi/2$ and π because that would correspond to \vec{dA} pointing into the hemisphere that is not visible to us. So θ runs from zero to $\pi/2$.

We now have

$$\langle \vec{dA} \cdot \hat{z} \rangle = \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{1}{2}. \quad (2)$$

(Notice that this integral is really just the differential projection of the solid's face onto the hemisphere we are sitting at the apex of, normalized (divided by) the surface area of the hemisphere.)

Here, the picture is that the orientation of the solid's face is changing while the observer's position is held fixed. In this picture, we are working in the rest frame of the observer. As we noted in the Introduction, it is equivalent to consider the solid's face fixed and vary the observer's position with respect to it – in other words, to work in the rest frame of the solid. The only quantity of physical significance is the relative angle between the line of sight and the oriented unit area vector \vec{dA} .

Taking into account the additional factor of $1/2$ because \vec{dA} only points into the hemisphere visible to us half the time, we have

$$\langle A_{\text{proj, 3-d}} \rangle = \int_S \langle \vec{dA} \cdot \hat{z} \rangle dS = \frac{1}{4} A_S, \quad (3)$$

where S denotes an integral over the surface of the solid and A_S is the solid's surface area. So this is the result for three spatial dimensions.

Before we move on, note that, although we derived equation (2) in the rest frame of the observer, working in the rest frame of the solid will prove more fruitful for generalizing the theorem to higher dimensions. For notice that, in this picture, as the observer's position varies (at fixed distance from the solid's face), it describes a hemisphere. So we simply wish to find the normalized (divided by the surface area of the hemisphere) projection of the solid's face onto this hemisphere. But the projection operator, or dot product, is commutative, so this is just the projection of the hemisphere onto the solid's face – which is simply a disc bounded by the circle created where the hemisphere intersects the plane of the solid's face.

As noted, to normalize, we divide by the surface area of the hemisphere. We also multiply by the factor of $1/2$ because only half the time is the face oriented into our hemisphere – or, equivalently, we could normalize by dividing by the surface area of the sphere, since that is twice that of the hemisphere.

3 Proof of method for higher dimensions

As we pointed out in the previous section, for the 3-d case, considering the rest frame of the solid's face and projecting onto the hemisphere defined by the different possible orientations of an observer at fixed distance is a fruitful way to calculate the average projected area. As we also pointed out, this projection can be easily evaluated by finding the normalized projection of the hemisphere onto the solid's face. Both of these ideas are also valid in arbitrary higher dimensions.

In higher dimensions, we simply consider the hyperhemisphere generated by varying the observer's orientation (at fixed distance) with respect to the solid, and look for the normalized projection of the solid's hypersurface onto it. This projection is just equal to the normalized projection of the hyperhemisphere onto the hypersurface, which bounds it (imagine the hemisphere as an upside-down bowl and the solid's face as the table it's on). And the projection of a hyperhemisphere of dimension d onto the bounding hypersurface in d dimensions is simply the hypersphere of dimension $d - 1$. So we see that

$$\langle dA_{\text{proj}, d} \rangle = \frac{V_{d-1}}{2S_{H, d}}, \quad (4)$$

where V_{d-1} is the volume of the unit hypersphere of dimension $d - 1$ and $S_{H, d}$ its surface area (the factor of $1/2$ is because the solid is only pointing into our hemisphere half the time). To continue the upside-down bowl analogy, here V_{d-1} is the area of the disc defined by the bowl's edge on the tabletop, and $S_{H, d}$ is just the normalization factor.

4 Calculation in higher dimensions

Here, we use the observation of the previous section to derive the analog of equation (3) for dimensions other than 3. As equation (4) shows, the differential

average projected area of a d -dimensional convex solid is just the volume of the unit $d-1$ -sphere normalized by double the surface area of the unit d -hemisphere. Hence we seek the factor $k(d)$ relating $\langle A_{\text{proj}, d} \rangle$ to A_S , the surface area of the convex solid. As we will see below, it is just V_{d-1}/S_d , where V_{d-1} is the volume of the unit $d-1$ -sphere and S_d is the surface area of the unit d -sphere. We have

$$\langle A_{\text{proj}, d} \rangle = \int_S \langle dA_{\text{proj}, d} \rangle dS = \frac{V_{d-1}}{S_d} A_S = k(d) A_S, \quad (5)$$

where the second equality is from using equation (4) for $\langle dA_{\text{proj}, d} \rangle$ and replacing $2S_{H, d}$ with S_d . We can check that the result is correct for 3-d: with $d=3$, $V_{d-1} = V_2 = \pi$ and $S_d = 4\pi$.

Now, for a sphere,

$$V_{d-1} = \frac{S_{d-1}}{d-1}, \quad (6)$$

and

$$S_{d-1} = \frac{2\pi^{d/2-1/2}}{\Gamma(\frac{1}{2}d - \frac{1}{2})} = \frac{1}{M_d \sqrt{\pi}} S_d \quad (7)$$

(Hypersphere, Wolfram), where we have defined

$$M_d = \frac{\Gamma(\frac{1}{2}(d-1))}{\Gamma(\frac{1}{2}d)}. \quad (8)$$

We can thus see that

$$k(d) = \frac{1}{\sqrt{\pi}(d-1)M_d}. \quad (9)$$

Equation (9) is easily evaluated by e.g. Mathematica; we provide a table of the first thirty-two values, as well as a plot illustrating the fall-off in $k(d)$ with increasing d .

It is possible that measurements of this factor could tell us something about the number of dimensions we live in. String theory suggests we may live in 9 (supersymmetric theories), 10 (M-theory), or 25 spatial dimensions (bosonic string theory); for reviews, see Kiritsis 1998, Polchinski 1998, or Schwarz 2000. However, in string theory these extra dimensions are often constrained to be on very small scales – for instance, in Kaluza-Klein theory, the extra dimension must be smaller than 10^{-16} cm (Yagi et al. 2011), and in Randall-Sundrum models (4 spatial dimensions, see Randall & Sundrum 1999 I and II), constraints are on the order of micro (10^{-6}) meters (Kapner et al. 2007). It is therefore not clear, and conceivably a topic of further research, whether extra dimensions would have effects measurable through a value of $k(d)$ different from $k(3) = 1/4$.

Finally, it is interesting to consider the limiting behavior of $k(d)$ as d tends to infinity. As might be intuited from the plot, $k(d) \rightarrow 0$ as $d \rightarrow \infty$. To prove

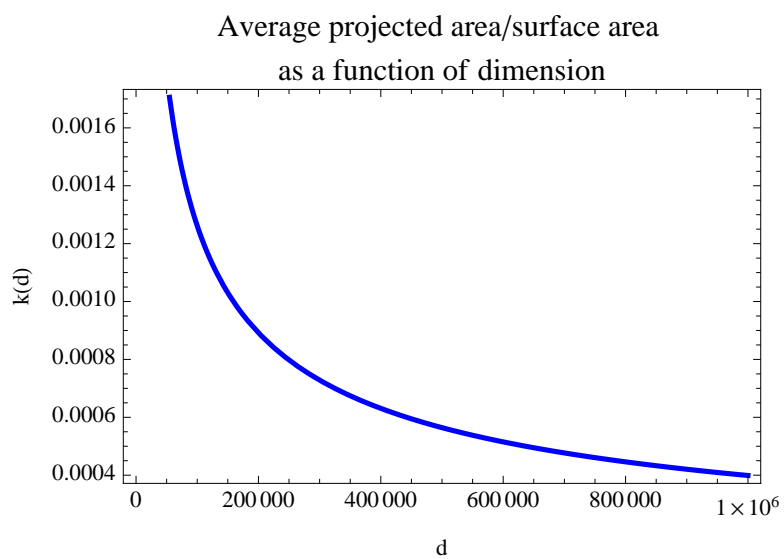


Figure 1: This plot shows that $k(d)$ falls off as d increases; to leading order $k(d) \propto 1/\sqrt{d}$, as equation (10) indicates.

Table 1: Ratio of average projected area to surface area ($k(d)$) as a function of dimension d

d	$k(d)$	d	$k(d)$
2	.318	18	.095
3	.250	19	.093
4	.212	20	.090
5	.1875	21	.088
6	.170	22	.086
7	.156	23	.084
8	.146	24	.082
9	.137	25	.081
10	.129	26	.079
11	.123	27	.077
12	.118	28	.076
13	.113	29	.075
14	.109	30	.073
15	.105	31	.072
16	.101	32	.071
17	.098	33	.070

this, we consider a series for $k(d)$ about $d = \infty$, with $x \equiv 1/d$:

$$k(d) = \frac{1}{\sqrt{2\pi}} \left\{ x^{1/2} + \frac{x^{3/2}}{4} + \frac{x^{5/2}}{32} - \frac{5x^{7/2}}{128} - \frac{21x^{9/2}}{2048} + \mathcal{O}\left(x^{11/2}\right) \right\}. \quad (10)$$

This series can be plotted against the exact values, and is accurate to one part in ten thousand even for d as low as five.

5 A recursion relation for $k(d)$

We can find a simple recursion relation for $k(d+1)$ in terms of $k(d)$. Observe that

$$M_d M_{d+1} = \frac{\Gamma(y)}{\Gamma(y+1)}, \quad (11)$$

where $y = \frac{1}{2}(d-1)$. Using the identity that $\Gamma(1+y) = y\Gamma(y)$, we have

$$M_d M_{d+1} = \frac{2}{d-1}. \quad (12)$$

Plugging into the above using equation (9) to relate $k(d)$ and $k(d+1)$ to, respectively, $M(d)$ and $M(d+1)$, we obtain the simple recursion

$$k(d+1) = \frac{1}{2\pi dk(d)}. \quad (13)$$

One can easily check that, starting with $k(3) = 1/4$, this formula allows us to reproduce the results of equation (9), given in Table 1.

6 An explicit formula for $k(d)$

Using the recursion (equation (13)) and anchoring it at $k(2) = 1/\pi$, we can derive an explicit formula for $k(d)$, $d > 2$. Writing out the values of $k(d)$ for d from 3 to 10 allows us to observe that the following formulae hold for $d > 2$:

$$k(d) = \frac{1}{2} \frac{(d-2)!_O}{(d-1)!_E}, \quad d \text{ odd} \quad (14)$$

and

$$k(d) = \frac{1}{\pi} \frac{(d-2)!_E}{(d-1)!_O}, \quad d \text{ even} \quad (15)$$

where subscripts O and E denote, respectively, that the factorial is taken only over the odd or even numbers. For instance, $10!_E = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$, and $9!_O = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$. As this is non-standard notation, I present these formulae written in terms of products below as well (for $d > 2$).

$$k(d) = \frac{1}{2} \prod_{n=0}^{(d-3)/2} \frac{2n+1}{2n+2}, \quad d \text{ odd} \quad (16)$$

and

$$k(d) = \frac{1}{\pi} \prod_{n=0}^{(d-4)/2} \frac{2n+2}{2n+3}, \quad d \text{ even}. \quad (17)$$

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